

# Convergence of the conical Ricci flow on $S^2$ to a soliton \*

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## Abstract

In our previous work [PSSW], we showed that the Ricci flow on  $S^2$  whose initial metric has conical singularities  $\sum_j \beta_j [p_j]$  converges to a constant curvature metric with conic singularities (in the stable and semi-stable cases) or to a gradient shrinking soliton with conical singularities (in the unstable case). The purpose of this note is to show that in the unstable case, that is, the case where  $\beta_k > \beta'_k = \sum_{j < k} \beta_j$ , that the limiting metric is the unique shrinking soliton with cone singularity  $\beta_k [p_\infty] + \beta'_k [q_\infty]$ . This verifies the prediction made in [PSSW].

## 1 Introduction

Let  $g_{S^2}$  be the round metric on  $S^2$  and  $\omega_{S^2} = \sqrt{-1} g_{z\bar{z}} dz \wedge d\bar{z}$  the Kähler form, so that  $[\omega_{S^2}] = 2[p]$  for any  $p \in S^2$ . Let  $p_1, \dots, p_k \in S^2$  be a finite collection of points and  $\beta_1 \leq \beta_2 \leq \dots \leq \beta_k \in (0, 1)$ . Let  $\beta = \sum_{j=1}^k \beta_j [p_j]$ .

A smooth metric  $g$  on  $S^2 \setminus \{p_1, \dots, p_k\}$  is a cone metric on  $(S^2, \beta)$  if it can be written in the form

$$\omega = \frac{e^f}{\prod_j |\sigma_j|_{\omega_{S^2}}^{\beta_j}} \cdot \omega_{S^2}$$

for some bounded function  $f$  on  $S^2$  where  $\sigma_j$  is a section of  $K_{S^2}^{-1}$  such that  $[\sigma_j] = 2[p_j]$ .

A constant curvature metric on  $(S^2, \beta)$  is a metric  $\omega_\phi = \omega_{S^2} + i\partial\bar{\partial}\phi$  with the property

$$\omega_\phi = \frac{e^{-\gamma\phi}}{\prod_j |\sigma_j|_{\omega_{S^2}}^{\beta_j}} \cdot \omega_{S^2} \quad (1.1)$$

where  $\gamma = 1 - \frac{1}{2} \sum \beta_j$ .

An alternative form of (1.1) is

$$\text{Ric}(\omega_\phi) = \gamma\omega_\phi + \sum \beta_j [p_j] . \quad (1.2)$$

The Ricci flow is given by

$$e^{-\dot{\phi}} \omega_\phi = \frac{e^{-\gamma\phi}}{\prod_j |\sigma_j|_{\omega_{S^2}}^{\beta_j}} \cdot \omega_{S^2} \quad , \quad \phi(0) = \phi_0 \quad (1.3)$$

where  $\dot{\phi} = \partial_t \phi$ . An alternative form of (1.3) is

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\*Work supported in part by National Science Foundation grants DMS-12-66033, DMS-0847524 and DMS-0905873 and a Collaboration Grants for Mathematicians from Simons Foundation.

$$\partial_t g = -\text{Ric}(g) + \gamma g, \quad g(0) = g_0 \quad \text{a cone metric on } (S^2, \beta). \quad (1.4)$$

Here, and in all that follows, we assume  $\gamma > 0$ . We shall also assume that  $g_0$  is “regular” in the sense of [PSSW]. This means that  $g_0$  is smooth on  $S^2 \setminus \{p_1, \dots, p_k\}$  and that in a neighborhood of  $p_j$  there is a holomorphic coordinate  $z$  such that

$$g_0 = e^{u_j} \frac{dz \wedge d\bar{z}}{|z|^{2\beta_j}} \quad (1.5)$$

where

$$u_0, \Delta_0 u_0, \Delta_0(\Delta_0 u_0) \in C^{2,\alpha}(S^2, \beta) \cap W^{1,2}.$$

Here  $C^{2,\alpha}(S^2, \beta)$  is the Yin-Hölder space defined in [Y]. In particular, if  $u_0$  is harmonic in a neighborhood of  $p_j$ , then  $g_0$  is regular.

Let  $\beta'_k = \sum_{j < k} \beta_j$  and let

$$\beta_\infty = \beta_k[p_\infty] + \beta'_k[q_\infty]$$

where  $p_\infty$  and  $q_\infty$  are the north and south pole respectively. Then we say  $\beta$  is stable, semi-stable or unstable if  $\beta'_k > \beta_k$ ,  $\beta'_k = \beta_k$ , or  $\beta'_k < \beta_k$  respectively.

When  $\beta$  is stable it is known, by the work of [MRS], that the Ricci flow converges to the unique constant scalar metric on  $(S^2, \beta)$ . In [PSSW] we give a new proof of this result. We also show that in the semi-stable case, the Ricci flow converges to the unique constant scalar curvature metric on  $(S^2, \beta_\infty)$ .

We now assume that  $\beta$  is unstable and we let  $g_\infty$  the unique conic shrinking soliton on  $(S^2, \beta_\infty)$ . This means that  $g_{sol}$  (which is rotationally symmetric by uniqueness) satisfies the following equation on  $S^2 \setminus \{p_\infty, q_\infty\}$ :

$$R(g_{sol}) = \gamma + \Delta_{g_{sol}} \theta_{sol}, \quad \nabla_{g_{sol}}^2 \theta_{sol} = \frac{1}{2}(\Delta_{g_{sol}} \theta_{sol})g_{sol}, \quad \int_{S^2} e^{\theta_{sol}} dg_{sol} = 2 \quad (1.6)$$

for a unique  $\theta_{sol} \in C^0(S^2) \cap C^\infty(S^2 \setminus \{p_\infty, q_\infty\})$ . Here  $R(g_{sol})$  is the scalar curvature of  $g_{sol}$ .

We wish to prove the following:

**Theorem 1** *For any initial regular metric  $g_0$  on  $(S^2, \beta)$  the Ricci flow converges to  $g_\infty$ .*

Remark: In [PSSW] we proved that there is a partition  $\{1, 2, \dots, k\} = I \cup J$  into disjoint subsets with the following property. The flow (1.4) converges to the unique Kähler-Ricci soliton  $g_{I,J}$  with cone structure

$$\beta_{I,J} = \left( \sum_{i \in I} \beta_i \right) p_i + \left( \sum_{j \in J} \beta_j \right) p_j.$$

Thus the content of Theorem 1 is that  $I = \{p_k\}$  and  $J = \{p_1, \dots, p_{k-1}\}$ . In particular,  $p_k \rightarrow p_\infty$  and  $p_1, \dots, p_{k-1} \rightarrow q_\infty$  as  $t \rightarrow \infty$ .

## 2 The proof

Let  $\beta = \sum_{j=1}^k \beta_j [p_j]$  and  $g$  a cone metric on  $(S^2, \beta)$ . Let  $f \in C^0(S^2) \cap W^{1,2}(S^2)$ . We define the normalized  $W$ -functional for the pair  $(g, f)$  by the same expression as in the smooth case,

$$W(g, f) = \int_{S^2 \setminus \beta} \left( \frac{1}{2\gamma} (R + |\nabla f|^2) + f \right) \frac{e^{-f}}{4\pi\tau} dg. \quad (2.1)$$

We also define

$$\mu(g) = \inf_f \{ W(g, f) : \int_{S^2} e^{-f} dg = 2 \}.$$

Let  $\mu_1 = \max\{\mu(g_{I,J})\}$  and  $\mu_2 = \max\{\mu(g_{I,J}) : g_{I,J} \neq g_\infty\}$  where, as above,

$$g_\infty = g_{I,J}, \text{ where } I = \{p_k\} \text{ and } J = \{p_1, \dots, p_{k-1}\}$$

We shall need the following (Lemma 7.3) from [PSSW] which was proved by first showing  $\mu(g_t)$  is increasing along the Ricci flow, and then using the toric structure of  $g_{I,J}$  to compare  $\mu$  invariants.

**Lemma 1** *We have  $\mu_1 > \mu_2$ . Moreover, if there exists a regular cone metric  $\tilde{g}_0$  on  $(S^2, \beta)$  with the property  $\mu(\tilde{g}_0) > \mu_2$ , then the Ricci flow on  $(S^2, \beta)$  converges in the Gromov-Hausdorff  $C^\infty$  topology to  $g_\infty$  for any initial metric  $g_0$ . Thus  $(S^2, g_t) \rightarrow (S^2, g_\infty)$  as metric spaces in the Gromov-Hausdorff topology. Moreover, for any compact subset  $K \subseteq S^2 \setminus \{p_\infty, q_\infty\}$  there exists a family of diffeomorphisms  $f_t : S^2 \rightarrow S^2$  such that  $f_t^* g_t \rightarrow g_\infty$  in  $C^\infty(K)$ .*

To prove Theorem 1, we start by choosing coordinates on  $\mathbb{P}^1$  in such a way that  $p_\infty$  is the point at infinity and  $q_\infty$  is the origin in  $\mathbb{C}$ . We define  $g_\beta$  to be the conic metric on  $(S^2, \beta)$  whose Kähler form is given by

$$\omega_\beta = c(\beta) \frac{\chi(z) dz \wedge d\bar{z}}{\prod_{j=1}^{k-1} |z - p_j|^{2\beta_j}} + c(\beta) \frac{(1 - \chi(z)) dz \wedge d\bar{z}}{(1 + |z|^2)^{2-\beta_k}} = F_\beta \omega_{FS}. \quad (2.2)$$

Here  $\chi$  is smooth with compact support on  $\mathbb{C}$  and equal to one in a large ball  $B$  centered at  $0 \in \mathbb{C}$  which contains  $p_1, \dots, p_{k-1}$  zero on the ball  $2B$ . The constant  $c(\beta)$  is chosen so that  $\int dg_\beta = 2$ . Thus  $q_\infty = 0 \in \mathbb{C}$  and  $p_\infty = \infty \in \mathbb{P}^1$ .

We have

$$\text{Ric } g_\beta = \sum_{j=1}^{k-1} \beta_j [p_j] \text{ on the ball } B.$$

Note that  $c(\beta)$  is a continuous function of  $p_1, \dots, p_{k-1}$  and is thus bounded from above and away from zero provided  $p_1, \dots, p_{k-1}$  remain in a bounded subset of  $\mathbb{C}$ .

Let  $\rho(t) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be the map  $z \mapsto tz$  for  $0 \leq t \leq 1$  and  $\rho(p_j) = p_j(t)$ . Thus  $p_j(t) = tp_j$  for  $j < k$  and  $p_k(t) = p_\infty$ . Let  $\beta(t) = \sum_{j=1}^k \beta_j [p_j(t)]$ . Then there exists  $q > 1$  such that

$$\beta(t) \rightarrow \beta_\infty \text{ and } g_{\beta(t)} \rightarrow g_{\beta_\infty} \text{ in the GH topology and } F_{\beta(t)} \rightarrow F_{\beta_\infty} \text{ in } L^q. \quad (2.3)$$

Here and in the following, when we use notation such as  $W^{1,2}, L^p, \Delta$  etc., the background metric is always  $g_{FS}$  unless otherwise specified.

We wish to construct a family of conic metrics  $g_t$  on  $(S^2, \beta(t))$  such that

$$g_t \rightarrow g_\infty \quad (2.4)$$

in the Gromov-Hausdorff topology and  $\mu(g_t) \rightarrow \mu(g_\infty)$ . Since  $\mu(g_\infty) = \mu_1 > \mu_2$  we conclude that for  $t$  sufficiently large,  $\mu(g(t)) > \mu_2$ . For such a  $t$ , we let  $g_0 = \rho(t)^* g(t)$ . Then  $g_0$  is a conic metric on  $(S^2, \beta)$  with the property  $\mu(g_0) > \mu_2$ , and so Lemma 1 applies to give the desired conclusion.

To define  $g(t)$  we first write

$$g_\infty = e^{u_\infty} g_{\beta_\infty}$$

for some continuous function  $u_\infty$  which is smooth on  $S^2 \setminus \beta_\infty$ . Theorem 1.1 of Datar-Guo-Song-Wang [DGSW] shows  $u_\infty$  is a “smooth  $S^1$  invariant conic metric”. This implies the  $u_\infty$  is smooth on  $\mathbb{C} \setminus \{0\}$  and that there is a smooth  $S^1$  invariant function  $\tilde{u}_\infty$  on  $B$  with the property  $u_\infty(z) = \tilde{u}_\infty(w)$  where  $|w|^2 = |z|^{2(1-\beta'_k)}$ . In particular,  $u(z)$  has a Taylor expansion of the form

$$u_\infty(z) = a_0 + a_1 |z|^{2(1-\beta'_k)} + a_2 |z|^{4(1-\beta'_k)} + \dots + a_m |z|^{2m(1-\beta'_k)} + O(|z|^{2(m+1)(1-\beta'_k)}) .$$

In particular, there exist  $C > 0$  such that on  $B$

$$|u_\infty(z) - a_0| + |z \partial_z u_\infty| + |z^2 \partial_z \bar{\partial}_z u_\infty| \leq C |z|^{2-2\beta'_k} . \quad (2.5)$$

We would like to define  $g(t) := e^{u_\infty} g_{\beta(t)}$ . This would satisfy (2.4) but doesn't quite work since  $u_\infty$  is not  $C^2$  on the complement of  $\beta(t)$  so  $e^{u_\infty} g_{\beta(t)}$  is not a regular metric in the sense of [PSSW]. Instead we proceed as follows. Let  $\psi = 1 - \chi$  which is zero on  $B$  and 1 outside  $2B$ . Define

$$u_\infty(t, z) = a_0 + \psi(z/t)(u_\infty(z) - a_0) \text{ if } t > 0 .$$

Thus for each  $t$  we see  $u_\infty(t, z) \in C^\infty(S^2 \setminus \{p_\infty\})$  and  $u_\infty(t, z)$  is constant on the ball  $tB$  and hence constant in a neighborhood of  $p_1(t), \dots, p_{k-1}(t)$ . Also,

$$u_\infty(t, z) \rightarrow u_\infty(z) \text{ pointwise as } t \rightarrow 0 ,$$

$$u_\infty(t) \rightarrow u_\infty \text{ and } \Delta_{g_{FS}} u_\infty(t) \rightarrow \Delta_{g_{FS}} u_\infty \text{ uniformly on compact subsets of } S^2 \setminus \beta_\infty .$$

Define

$$g(t) = e^{u_\infty(t)} g_{\beta(t)} .$$

We see that for each  $t > 0$  that  $g(t)$  satisfies (1.5) with  $u_j$  harmonic in a neighborhood of  $p_j$ . In particular,  $g(t)$  is a regular metric. Moreover,

$$\partial_z \bar{\partial}_z u_\infty(t, z) = \psi'(z/t) \frac{1}{t} [\partial_z u_\infty + \bar{\partial}_z u_\infty] + \psi''(z/t) \frac{1}{t^2} (u_\infty(z) - a_0) + \psi(z/t) \partial_z \bar{\partial}_z u_\infty .$$

Since  $|t| \geq c|z|$  when the right side is non-zero, we conclude from (2.5) that on  $B$

$$|\partial_z \bar{\partial}_z u_\infty(t, z)| \leq \frac{C}{|z|^{2\beta'_k}}$$

for some  $C > 0$  which is independent of  $t$ . We conclude that there exists  $q > 1$  such that  $\|R(g(t))\|_{L^q} \leq C$  for all  $t > 0$ . Moreover, decreasing  $q$  slightly if necessary,  $\|R(t)\|_{L^q} \rightarrow \|R(g_\infty)\|_{L^q}$ .

In general, if  $(X_t, g_t) \rightarrow (X, g_\infty)$  is any Gromov-Hausdorff limit of smooth manifolds, we know ([Chow], Lemma 6.28) that

$$\mu(g_\infty) \geq \limsup \mu(g_t)$$

Thus our goal is to show  $\lim_{T \rightarrow \infty} \inf_{t \geq T} \mu(g(t)) \geq \mu(g_\infty)$ . Assume not. Then there exist  $\delta > 0$  and a sequence  $t_j \rightarrow \infty$  such that

$$\mu(g_j) = \mu(g(t_j)) \leq \mu(g_\infty) - \delta \quad (2.6)$$

Thus for each  $j$  there is a positive function  $\Phi_j = e^{-f_j/2}$  such that  $\|\Phi_j\|_{L^2(g_j)} = 1$  and

$$W(g_j, f_j) = \int_{S^2 \setminus \beta(t_j)} \left( \frac{2}{\gamma} |\nabla_j \Phi_j|^2 - \Phi_j^2 \frac{R_j}{2\gamma} - \Phi_j^2 \log \Phi_j^2 \right) dg_j = \mu(g_j) \leq \mu(g_\infty) - \delta \quad (2.7)$$

where  $\gamma = 1 - \frac{1}{2} \sum_j \beta_j$ .

**Lemma 2** *We have the following bounds.*

1. *The  $\Phi_j$  are uniformly bounded in  $W^{1,2}$  that is, there exists  $C > 0$  such that*

$$\int \Phi_j^2 dg_{FS} + \int \partial \Phi_j \wedge \bar{\partial} \Phi_j \leq C \text{ for all } j$$

2. *There exists  $q > 1$  such that  $\Delta u_j \rightarrow \Delta u_\infty$  in  $L^q$ , that is*

$$\lim_{j \rightarrow \infty} \int_{S^2} |\Delta_{g_{FS}} u_j - \Delta_{g_{FS}} u_\infty|^q dg_{FS} = 0$$

We postpone the proof for the moment and show how the lemma leads to a contradiction.

Part (1) implies there exists  $\Phi_\infty \in W^{1,2}$  such that  $\Phi_j \rightharpoonup \Phi_\infty$  that is,  $\Phi_j$  converges weakly to  $\Phi_\infty$  in  $W^{1,2}$ . Since  $W^{1,2} \hookrightarrow L^p$  is a compact imbedding for all  $p > 1$ , we see that after passing to a subsequence,  $\Phi_j \rightarrow \Phi_\infty$  in  $L^p$  for all  $p > 1$ . Thus  $\|\Phi_j\|_{L^p} \leq C_p$  for all  $j$  and  $\Phi_j^2 \rightarrow \Phi_\infty^2$  in  $L^p$  for all  $p$ .

We claim

$$\int \Phi_j^2 R_j dg_j = \int \Phi_j^2 (\Delta u_j + \gamma) dg_{FS} \rightarrow \int \Phi_\infty^2 (\Delta u_\infty + \gamma) dg_{FS} = \int \Phi_\infty^2 R_\infty dg_j \quad (2.8)$$

$$1 = \int \Phi_j^2 dg_j = \int \Phi_j^2 F_{\beta_j} dg_{FS} \rightarrow \int \Phi_\infty^2 F_{\beta_\infty} dg_{FS} = \int \Phi_\infty^2 dg_\infty = 1 \quad (2.9)$$

$$\int \Phi_j^2 \log \Phi_j^2 dg_j = \int \Phi_j^2 \log \Phi_j^2 F_{\beta_j} dg_{FS} \rightarrow \int \Phi_\infty^2 \log \Phi_\infty^2 F_{\beta_\infty} dg_{FS} = \int \Phi_\infty^2 \log \Phi_\infty^2 dg_j \quad (2.10)$$

$$\liminf_j \int |\nabla_j \Phi_j|^2 dg_j \geq \int |\nabla_\infty \Phi_\infty|^2 dg_\infty \quad (2.11)$$

To prove (2.8) we note that  $\Delta u_j \rightarrow \Delta u_\infty$  in  $L^q$  and  $\Phi_j^2 \rightarrow \Phi_\infty^2$  in  $L^p$  for all  $p$ . Similarly (2.9) follows from the fact that  $F_{\beta_j} \rightarrow F_{\beta_\infty}$  in  $L^q$  for some  $q = q(\beta)$ .

To prove (2.10) we need only show  $\Phi_j^2 \log \Phi_j^2 \rightarrow \Phi_\infty^2 \log \Phi_\infty^2$  in  $L^p$  for all  $p$ . To see this, first note that if  $x, y > 0$  there exists  $\theta$  between  $x$  and  $y$  such that

$$|x^2 \log x^2 - y^2 \log y^2| = |4\theta \log \theta + 2\theta| \cdot |x - y| \leq C_\delta(1 + |x|^2 + |y|^2) \cdot |x - y| \quad (2.12)$$

by the mean value theorem (c.f. [R]). Now substitute  $x = \Phi_j$  and  $y = \Phi_\infty$  and apply Hölder's inequality.

Finally (2.11), which is equivalent to

$$\liminf_j \int \partial \Phi_j \wedge \bar{\partial} \Phi_j \geq \int \partial \Phi_\infty \wedge \bar{\partial} \Phi_\infty, \quad (2.13)$$

Since  $\Phi_j \rightarrow \Phi_\infty$  in  $W^{1,2}$  we know

$$\liminf_j \|\Phi_j\|_{W^{1,2}} \geq \|\Phi_\infty\|_{W^{1,2}} \quad (2.14)$$

But  $\Phi_j \rightarrow \Phi_\infty$  strongly in  $L^2(g_{FS})$ . Thus (2.13) follows from (2.14).

Taking the  $\lim_{j \rightarrow \infty}$  of both sides of (2.7) and applying (2.8), (2.9), (2.10) and (2.11), we obtain

$$\int \left[ \frac{2}{\gamma} |\nabla_j \Phi_\infty|^2 - \Phi_\infty^2 \frac{R_\infty}{2\gamma} - \Phi_\infty^2 \log \Phi_\infty^2 \right] e^{u(t_\infty)} g_{\beta_\infty} \leq \mu(g_\infty) - \delta \quad (2.15)$$

which contradicts the definition of  $\mu(g_\infty)$ .

Thus we have reduced the proof of Proposition 1 to the proof of the lemma.

To prove the lemma, note that (2.7) implies

$$\int \partial \Phi_j \wedge \bar{\partial} \Phi_j \leq C \|\Phi_j\|_{L^q}^2$$

for some  $q > 2$ . On the other hand

$$\|\Phi_j\|_{L^q}^2 - C_{q'} \|\Phi_j\|_{L^2}^2 \leq C_{q'} \int \partial \Phi_j \wedge \bar{\partial} \Phi_j$$

for any  $q' > q$ . This implies  $\|\Phi_j\|_{L^q} \leq C$  and hence  $\int \partial \Phi_j \wedge \bar{\partial} \Phi_j \leq C$ .

This concludes the proof first part of the lemma. The second follows from the fact that  $\Delta u_j \rightarrow \Delta u_\infty$  pointwise almost everywhere and  $\|\Delta u_j\|_{L^q}$  is uniformly bounded for some  $q = q(\beta) > 1$ .

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